

DIVISOR PROBLEM IN ARITHMETIC PROGRESSIONS MODULO A PRIME POWER

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ABSTRACT. We obtain an asymptotic formula for the average value of the divisor function over the integers $n \leq x$ in an arithmetic progression $n \equiv a \pmod{q}$, where $q = p^k$ for a prime $p \geq 3$ and a sufficiently large integer k . In particular, we break the classical barrier $q \leq x^{2/3}$ for such formulas, and generalise a recent result of R. Khan (2015), making it uniform in k .

1. INTRODUCTION

1.1. Background. For a positive integer n , let $d(n)$ be the classical divisor function, which is the number of divisors of n . Let a and q be integers with $q \geq 1$ and $\gcd(a, q) = 1$. For $X \geq 2$, define

$$D(X; q, a) := \sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} d(n).$$

and also

$$E(X; q, a) := D(X; q, a) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \\ \gcd(n, q) = 1}} d(n).$$

In unpublished works, it has been discovered independently by Selberg and Hooley that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for a sufficiently large X

$$(1.1) \quad |E(X; q, a)| \leq X^{1-\delta}/q$$

holds uniformly for $q \leq X^{2/3-\varepsilon}$. This follows from Weil bound for Kloosterman sums, see [16].

When q is large, there are various results on the average bound of $E(X; q, a)$. Fouvry [3, Corollary 5] has studied the average over q and shown that for any $\varepsilon > 0$ there exist some constant $c > 0$ such that

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for a sufficiently large X for any $a \in \mathbb{Z}$ with $|a| \leq \exp(c\sqrt{\log X})$ we have

$$\sum_{\substack{X^{2/3+\varepsilon} \leq q \leq X^{1-\varepsilon} \\ \gcd(q,a)=1}} |E(X; q, a)| \leq X \exp(-c\sqrt{\log X})$$

Banks, Heath-Brown and Shparlinski [1] have considered the average over a and proved that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for a sufficiently large X

$$\sum_{\substack{1 \leq a \leq q \\ \gcd(a,q)=1}} |E(X; q, a)| \leq X^{1-\delta}$$

holds uniformly for $q < X^{1-\varepsilon}$. For other examples, see [2, 4, 6, 7, 15].

Irving [8] first has broken through the range given by Weil bound (see [9, Corollary 11.12]) for some special individual modulus q and proved that, for any $\varpi, \varrho > 0$ satisfying $246\varpi + 18\varrho < 1$, there exists some $\delta > 0$, depending only on ϖ and ϱ such that (1.1) holds uniformly for any x^ϱ -smooth, squarefree moduli $q \leq X^{2/3+\varpi}$. Khan [10] has considered another important case: the prime power moduli and proved that for a fixed integer $k \geq 7$, there exists some constant $\rho > 0$, depending only on k , such that (1.1) holds uniformly for $X^{2/3-\rho} < q < X^{2/3+\rho}$ with $q = p^k$, where p is a sufficiently large prime number.

1.2. Our results. In this paper, we focus on the prime power moduli case.

Before we formulate our result we need to recall that the notations $U \ll V$ and $U = O(V)$, are equivalent to $|U| \leq cV$ for some constant $c > 0$. We write \ll_ρ and O_ρ to indicate that this constant may depend on the parameter ρ .

Theorem 1.1. *There exist absolute constants $k_0 \geq 1$ and $\sigma > 0$ such that*

$$E(X; q, a) \ll \frac{X}{q^{1+\sigma}}$$

holds uniformly for $q \leq X^{2/3+\sigma}$ with $q = p^k$ for an odd prime p and integer $k \geq k_0$.

The key in our proof of Theorem 1.1 is the following average estimate for Kloosterman sums

$$S(n, a; q) := \sum_{b \bmod q}^* e\left(\frac{nb + a\bar{b}}{q}\right),$$

with prime power moduli, where $\gcd(a, q) = 1$ and \sum^* means summing over reduced residue classes. The proof borrows from some ideas from [17, 18] reworked and adjusted to the case which is relevant to Kloosterman sums.

Theorem 1.2. *For any $q^\lambda \leq N \leq q$ with $\lambda > 0$, there exist constants k_0 and $\tau > 0$, depending only on λ such that*

$$\sum_{1 \leq n \leq N} S(n, a; q) \ll_\lambda N q^{1/2-\tau}$$

holds uniformly for any integers a satisfying $\gcd(a, p) = 1$ and any $q = p^k$ with p an odd prime, $k \geq k_0$.

Using that for any integers a , m and n with $\gcd(m, p) = 1$ we have

$$S(mn, a; q) = S(a, mn; q) = S(n, am; q).$$

Now we reformulate Theorem 1.2 in the form in which we apply it in the proof of Theorem 1.1:

Corollary 1.3. *For any $q^\lambda \leq N \leq q$ with $\lambda > 0$, there exist constants k_0 and $\tau > 0$, depending only on λ such that*

$$\sum_{1 \leq n \leq N} S(mn, a; q) = \sum_{1 \leq n \leq N} S(a, mn; q) \ll_\lambda N q^{1/2-\tau}$$

holds uniformly for any integers m, a satisfying $\gcd(ma, p) = 1$ and any $q = p^k$ with p an odd prime, $k \geq k_0$.

Remark 1.4. *Comparing with the result of Khan [10], in which the condition k fixed and p sufficiently large is required, Theorem 1.1 gives a uniform result for all modulus of the type $q = p^k$ with p an odd prime and k sufficiently large.*

Remark 1.5. *In Theorem 1.2, since $\lambda > 0$ can be taken arbitrary small, our result shows that Weil bound for sums of Kloosterman sums can be improved on average over a very short interval for prime power modulus.*

1.3. Notation. As usual, \mathbb{N} , \mathbb{Z} , \mathbb{R} and \mathbb{Z}_p are the set of natural numbers, integers, real numbers and p -adic integers, respectively. We use $e(x)$ to denote $e^{2\pi i x}$ and $[x]$ to denote the largest integer not exceeding x . For a prime number p and any $n \in \mathbb{Z}$, $p^r \parallel n$ means $p^r | n$ and $p^{r+1} \nmid n$.

For a p -adic integer $\alpha \in \mathbb{Z}_p$, denote its p -adic order as $v_p(\alpha)$. For a polynomial $f(x)$ with integer coefficients, denote $\text{ord}_p f$ as the p -adic order of the largest common divisor of all the coefficients of f (that is, the largest power of p which divides all the coefficients of f).

2. PROOF OF THEOREM 1.1

We now assume that Theorem 1.2 holds, and then prove it in Section 3. In particular, here we use Corollary 1.3.

By the definition of $d(n)$, we have

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} d(n) = \sum_{\substack{uv \leq x \\ uv \equiv a \pmod q}} 1.$$

Let $\varepsilon > 0$ be sufficiently small and $\Delta = 1 + x^{-2\varepsilon}$. Suppose U, V are parameters of the form Δ^i and Δ^j for $i, j \geq 0$, separately. Then we have

$$\sum_{\substack{uv \leq x \\ uv \equiv a \pmod q}} 1 = \sum_{U, V} \sum_{\substack{uv \leq x \\ uv \equiv a \pmod q \\ U < u \leq \Delta U \\ V < v \leq \Delta V}} 1,$$

where $\sum_{U, V}$ ranges over all the pairs $U = \Delta^i$, $V = \Delta^j$ satisfying $UV \leq x$. The number of these pairs is at most $O(x^{4\varepsilon} \log^2 x)$. Removing the condition $uv \leq x$ in the inner sum on the right hand side,

$$\sum_{\substack{uv \leq x \\ uv \equiv a \pmod q}} 1 = \sum_{U, V} \sum_{\substack{uv \equiv a \pmod q \\ U < u \leq \Delta U \\ V < v \leq \Delta V}} 1 + O\left(\sum_{\substack{x < n \leq x\Delta^2 \\ n \equiv a \pmod q}} d(n)\right).$$

It is obvious that the error term is $O_\varepsilon\left(\frac{x^{1-\varepsilon}}{q}\right)$. We can restrict the range of the sum over in the first term to $x^{1-2\varepsilon} < UV \leq x$ up to an acceptable error term, since

$$\sum_{\substack{U, V \\ UV \leq x^{1-2\varepsilon}}} \sum_{\substack{uv \equiv a \pmod q \\ U < u \leq \Delta U \\ V < v \leq \Delta V}} 1 \leq \sum_{\substack{n \leq x^{1-2\varepsilon} \Delta^2 \\ n \equiv a \pmod q}} d(n) \ll_\varepsilon \frac{x^{1-\varepsilon}}{q}.$$

Hence we have

$$(2.1) \quad \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} d(n) = \sum_{\substack{U, V \\ x^{1-2\varepsilon} \leq UV \leq x}} \sum_{\substack{uv \equiv a \pmod q \\ U < u \leq \Delta U \\ V < v \leq \Delta V}} 1 + O_\varepsilon\left(\frac{x^{1-\varepsilon}}{q}\right).$$

Now we smooth the inner sum over u and v . Suppose f and g are smooth functions and compactly supported on the interval $[1, \Delta]$ with derivatives satisfying

$$f^{(j)} \ll_j x^{6j\varepsilon} \quad \text{and} \quad g^{(j)} \ll_j x^{6j\varepsilon} \quad \text{for any } j \geq 0$$

and f, g equals 1 in the interval $[1 + x^{-6\varepsilon}, \Delta - x^{-6\varepsilon}]$. Replacing the 1 in the inner sum on the right hand side of (2.1) by $f\left(\frac{u}{U}\right)g\left(\frac{v}{V}\right)$, it is easy to prove that the contribution of the error terms produced in this process can be absorbed by the O -term. Then we have

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} d(n) = \sum_{\substack{U, V \\ x^{1-2\varepsilon} \leq UV \leq x}} I(U, V; q, a) + O_\varepsilon\left(\frac{x^{1-\varepsilon}}{q}\right),$$

where $I(U, V; q, a)$ is defined by

$$I(U, V; q, a) := \sum_{\substack{u, v \\ uv \equiv a \pmod{q}}} f\left(\frac{u}{U}\right)g\left(\frac{v}{V}\right).$$

By a similar argument, we can get

$$\frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ \gcd(n, q) = 1}} d(n) = \frac{1}{\varphi(q)} \sum_{\substack{U, V \\ x^{1-2\varepsilon} \leq UV \leq x}} I(U, V) + O_\varepsilon\left(\frac{x^{1-\varepsilon}}{q}\right),$$

with $I(U, V)$ given by

$$I(U, V) := \sum_{\substack{u, v \\ \gcd(uv, q) = 1}} f\left(\frac{u}{U}\right)g\left(\frac{v}{V}\right).$$

Thus we have

$$\begin{aligned} & \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} d(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ \gcd(n, q) = 1}} d(n) \\ &= \sum_{\substack{U, V \\ x^{1-2\varepsilon} \leq UV \leq x}} \left(I(U, V; q, a) - \frac{1}{\varphi(q)} I(U, V) \right) + O_\varepsilon\left(\frac{x^{1-\varepsilon}}{q}\right). \end{aligned}$$

Now by the symmetry of U and V , we only need to prove

$$I(U, V; q, a) - \frac{1}{\varphi(q)} I(U, V) \ll_\varepsilon \frac{x^{1-\varepsilon}}{q}$$

for any U and V satisfying

$$x^{1-2\varepsilon} \leq UV \leq x \quad \text{and} \quad U \leq x^{1/2}.$$

Thus, we now fix U and V with this condition.

By the orthogonality of additive characters, we have

$$I(U, V; q, a) = \frac{1}{q} \sum_{h=1}^q e\left(\frac{-ah}{q}\right) \sum_{\substack{u, v \\ \gcd(u, q) = 1}} f\left(\frac{u}{U}\right)g\left(\frac{v}{V}\right) e\left(\frac{uvh}{q}\right).$$

Denote the term for $h = q$ by

$$\mathcal{M} := \frac{1}{q} \sum_{\substack{u \\ \gcd(u, q)=1}} f\left(\frac{u}{U}\right) \sum_v g\left(\frac{v}{V}\right).$$

By the definition of g , the inner sum over v is

$$\sum_{V < v \leq \Delta V} 1 + O(x^{-6\varepsilon} V) = (\Delta V - V) + O(1 + x^{-6\varepsilon} V),$$

which yields

$$(2.2) \quad \mathcal{M} = \frac{1}{q} \sum_{\substack{U < u \leq \Delta U \\ \gcd(u, q)=1}} f\left(\frac{u}{U}\right) (\Delta V - V) + O_\varepsilon\left(\frac{U}{q} + \frac{x^{1-6\varepsilon}}{q}\right).$$

Similarly, we have

$$\frac{1}{\varphi(q)} I(U, V) = \frac{1}{\varphi(q)} \sum_{\substack{U < u \leq \Delta U \\ \gcd(u, q)=1}} f\left(\frac{u}{U}\right) \sum_{\substack{V < v \leq \Delta V \\ \gcd(v, q)=1}} 1 + O_\varepsilon\left(\frac{x^{1-6\varepsilon}}{q}\right).$$

To remove the condition $\gcd(v, q) = 1$ in the sum over v , we use the formula

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and get

$$\sum_{\substack{V < v \leq \Delta V \\ \gcd(v, q)=1}} 1 = \sum_{d|q} \mu(d) \sum_{\substack{V < v \leq \Delta V \\ d|v}} 1.$$

It follows that

$$\sum_{\substack{V < v \leq \Delta V \\ \gcd(v, q)=1}} 1 = \frac{\varphi(q)}{q} (\Delta V - V) + O_\varepsilon(q^\varepsilon),$$

where we used

$$\sum_{d|q} \frac{\mu(d)}{d} = \frac{\varphi(q)}{q}.$$

Thus we obtain

$$(2.3) \quad \begin{aligned} \frac{1}{\varphi(q)} I(U, V) &= \frac{1}{q} \sum_{\substack{U < u \leq \Delta U \\ \gcd(u, q)=1}} f\left(\frac{u}{U}\right) (\Delta V - V) \\ &\quad + O_\varepsilon\left(\frac{U q^\varepsilon}{\varphi(q)} + \frac{x^{1-6\varepsilon}}{q}\right). \end{aligned}$$

Recall that $U \leq x^{1/2}$, then for sufficiently small ε , we get

$$\mathcal{M} - \frac{1}{\varphi(q)} I(U, V) \ll_{\varepsilon} \frac{x^{1-6\varepsilon}}{q}$$

from (2.2) and (2.3). Now we only need to estimate the sum

$$\mathcal{E} := \frac{1}{q} \sum_{h=1}^{q-1} e\left(\frac{-ah}{q}\right) \sum_{\substack{u,v=-\infty \\ \gcd(u,q)=1}}^{\infty} f\left(\frac{u}{U}\right) g\left(\frac{v}{V}\right) e\left(\frac{uvh}{q}\right)$$

and show that there exists an absolute constant $\sigma > 0$ such that

$$(2.4) \quad \mathcal{E} \ll \frac{x}{q^{1+\sigma}}$$

holds uniformly for $q \leq x^{2/3+\sigma}$. Note that since the functions f and g are compactly supported, the sum over u and v is actually finite.

Noting $q = p^k$ with p an odd prime, write

$$\mathcal{E} = \frac{1}{q} \sum_{0 \leq r < k} \sum_{\substack{1 \leq h \leq p^k \\ p^r \parallel h}} e\left(\frac{-ah}{q}\right) \sum_{\substack{u,v=-\infty \\ \gcd(u,q)=1}}^{\infty} f\left(\frac{u}{U}\right) g\left(\frac{v}{V}\right) e\left(\frac{uvh}{q}\right).$$

It follows that

$$\mathcal{E} = \frac{1}{q} \sum_{0 \leq r < k} \sum_{b \bmod p^{k-r}}^* e\left(\frac{-ab}{p^{k-r}}\right) \sum_{\substack{u,v=-\infty \\ \gcd(u,q)=1}}^{\infty} f\left(\frac{u}{U}\right) g\left(\frac{v}{V}\right) e\left(\frac{uvb}{p^{k-r}}\right).$$

The inner sum for u, v can be written as

$$\mathcal{F} := \sum_{\substack{s,t \bmod p^{k-r} \\ \gcd(t,p)=1}} e\left(\frac{stb}{p^{k-r}}\right) \sum_{u \equiv t \bmod (p^{k-r})} f\left(\frac{u}{U}\right) \sum_{v \equiv s \bmod (p^{k-r})} g\left(\frac{v}{V}\right).$$

Applying Poisson summation (see [5, Lemma 2.1]), it equals to

$$\frac{UV}{p^{2(k-r)}} \sum_{\substack{s,t \bmod p^{k-r} \\ \gcd(t,p)=1}} e\left(\frac{stb}{p^{k-r}}\right) \sum_{m,n} e\left(\frac{sn+tm}{p^{k-r}}\right) \widehat{f}\left(\frac{nU}{p^{k-r}}\right) \widehat{g}\left(\frac{mV}{p^{k-r}}\right).$$

Summing over s , we get

$$\mathcal{F} = \frac{UV}{p^{k-r}} \sum_{\substack{m,n \\ \gcd(n,p)=1}} e\left(-\frac{mn\bar{b}}{p^{k-r}}\right) \widehat{f}\left(\frac{nU}{p^{k-r}}\right) \widehat{g}\left(\frac{mV}{p^{k-r}}\right),$$

which gives

$$\mathcal{E} = \sum_{0 \leq r < k} \frac{UV}{p^{2k-r}} \sum_{\substack{m, n \\ \gcd(n, p)=1}} \hat{f}\left(\frac{nU}{p^{k-r}}\right) \hat{g}\left(\frac{mV}{p^{k-r}}\right) S(a, mn; p^{k-r}).$$

By partial integration, the sums over m and n can be restricted to

$$(2.5) \quad |n| \leq \frac{x^\varepsilon p^{k-r}}{U}, \quad |m| \leq \frac{x^\varepsilon p^{k-r}}{V},$$

up to an error term $O(x^{-100})$.

Break the sum over r into two sums

$$(2.6) \quad \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\begin{aligned} \mathcal{E}_1 &= \sum_{0 \leq r < k/8} \frac{UV}{p^{2k-r}} \sum_{\substack{m, n \\ \gcd(n, p)=1}} \hat{f}\left(\frac{nU}{p^{k-r}}\right) \hat{g}\left(\frac{mV}{p^{k-r}}\right) S(a, mn; p^{k-r}), \\ \mathcal{E}_2 &= \sum_{k/8 \leq r < k} \frac{UV}{p^{2k-r}} \sum_{\substack{m, n \\ \gcd(n, p)=1}} \hat{f}\left(\frac{nU}{p^{k-r}}\right) \hat{g}\left(\frac{mV}{p^{k-r}}\right) S(a, mn; p^{k-r}). \end{aligned}$$

For large r , we apply the Weil bound for Kloosterman sums (see [9, Corollary 11.12]) and derive

$$(2.7) \quad \mathcal{E}_2 \ll x^{2\varepsilon} p^{k/2} \sum_{k/8 \leq r < k} p^{-3r/2} \ll x^{2\varepsilon} p^{5k/16},$$

which is small enough.

Now we only need to bound \mathcal{E}_1 . Note that for $\gcd(a, p) = 1$ and $p^j \parallel m$,

$$S(a, m; p^k) = \begin{cases} \mu(p^k), & \text{if } j \geq k, \\ 0, & \text{if } 0 < j < k. \end{cases}$$

We have

$$\mathcal{E}_1 = \sum_{0 \leq r < k/8} \frac{UV}{p^{2k-r}} \sum_{\substack{m, n \\ \gcd(mn, p)=1}} \hat{f}\left(\frac{nU}{p^{k-r}}\right) \hat{g}\left(\frac{mV}{p^{k-r}}\right) S(a, mn; p^{k-r}).$$

Our cancellation comes from the sum over n . By (2.5), we only deal with

$$\mathcal{G} := \sum_{\substack{1 \leq n \leq \frac{x^\varepsilon p^{k-r}}{U} \\ \gcd(mn, p)=1}} \hat{f}\left(\frac{nU}{p^{k-r}}\right) S(a, mn; p^{k-r}).$$

The contribution of the part $n \leq -1$ can be treated similarly. Denote the sums over $1 \leq n \leq q^{1/10}$ and $n > q^{1/10}$ by $\mathcal{G}_{n \leq q^{1/10}}$ and

$\mathcal{G}_{n>q^{1/10}}$, respectively. Then Weil bound for Kloosterman sums (see [9, Corollary 11.12]) gives

$$\mathcal{G}_{n\leq q^{1/10}} \ll q^{1/10}(k-r+1)p^{\frac{k-r}{2}}.$$

Denote the contribution of $\mathcal{G}_{n\leq q^{1/10}}$ to \mathcal{E}_1 by \mathfrak{C}_1 , then we have

$$(2.8) \quad \mathfrak{C}_1 \ll (k+1)x^{1/2}q^{1/10} \sum_{0\leq r<k/8} p^{-\frac{k+r}{2}} \ll_{\varepsilon} x^{1/2}q^{-2/5}\mathcal{L}^2,$$

which is acceptable. For $\mathcal{G}_{n>q^{1/10}}$, it follows from partial summation that

$$\begin{aligned} \mathcal{G}_{n>q^{1/10}} &= \widehat{f}(q^{1/10}) \sum_{q^{1/10}<n\leq x^{\varepsilon}p^{k-r}/U} S(a, mn; p^{k-r}) \\ &\quad + \int_{q^{1/10}}^{\frac{x^{\varepsilon}p^{k-r}}{U}} \sum_{n\leq t} S(a, mn; p^{k-r}) \left(\widehat{f}\left(\frac{tU}{p^{k-r}}\right) \right)' dt. \end{aligned}$$

Note that $|\widehat{f}(t)| \leq 1$ for any $t \in \mathbb{R}$, then

$$\left(\widehat{f}\left(\frac{tU}{p^{k-r}}\right) \right)' \ll \frac{U}{p^{k-r}}.$$

Now by Corollary 1.3 there exists a constant $\rho > 0$ (which does not depend on ε), such that

$$\mathcal{G}_{n>q^{1/10}} \ll_{\varepsilon} \frac{x^{2\varepsilon}p^{k-r}q^{1/2-\rho}}{U}.$$

Let \mathfrak{C}_2 denote the contribution of $\mathcal{G}_{n>q^{1/10}}$ to \mathcal{E}_1 , then we have

$$(2.9) \quad \mathfrak{C}_2 \ll_{\varepsilon} \sum_{0\leq r<k/8} x^{3\varepsilon}q^{1/2-\rho}p^{-r} \ll_{\varepsilon} x^{3\varepsilon}q^{1/2-\rho}.$$

Since ε is arbitrary, combining (2.6), (2.7), (2.8) and (2.9) with (2.4), we complete the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

3.1. Preparations. We start with the following well-known elementary statement.

Lemma 3.1. *Let p be a prime number and $n \in \mathbb{N}$, then we have*

$$\text{ord}_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

We also need the following technical result.

Lemma 3.2. *For every integer $i \geq 1$, let*

$$\binom{1/2}{i} := \frac{1/2(1/2-1)\cdots(1/2-i+1)}{i!}$$

and $3 \leq u \leq i$, then we have

$$\nu_p \left(\binom{1/2}{i} \frac{i!}{(i-u)!} \right) \leq u \sum_{j=1}^{\infty} \frac{1}{p^j} + E(i, p),$$

with

$$|E(i, p)| \leq \frac{3 \log(2i)}{\log p}.$$

Proof. Noting that

$$\binom{1/2}{i} = \frac{(-1)^{i-1}(2i-3)!}{2^{2i-2}i!(i-2)!} \quad \text{for } i \geq 3,$$

we have

$$\binom{1/2}{i} \frac{i!}{(i-u)!} = (-1)^{i-1} \frac{(2i-3)!}{2^{2i-2}(i-2)!(i-u)!}$$

for $i \geq 3$, then by Lemma 3.1, we have

$$\nu_p \left(\frac{(2i-3)!}{(i-2)!(i-u)!} \right) = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{2i-3}{p^j} \right\rfloor - \left\lfloor \frac{i-2}{p^j} \right\rfloor - \left\lfloor \frac{i-u}{p^j} \right\rfloor \right).$$

Terms in the above sum vanish when $j > J$, where

$$J = \frac{\log(2i-3)}{\log p},$$

which yields

$$\nu_p \left(\frac{(2i-3)!}{(i-2)!(i-u)!} \right) = (u-1) \sum_{j=1}^J \frac{1}{p^j} + E(i, p),$$

with $|E(i, p)| \leq 3J \leq 3 \log(2i)/\log p$. Then the result follows from extending the range of the summation. \square

Lemma 3.3. *Let p be an odd prime and $k \geq 2$ be a positive integer. If $(a, p) = 1$ and $p|n$, then the Kloosterman sums $S(n, a; p^k) = 0$.*

Proof. By assumption, we may suppose $n = p^r m$ with $(m, p) = 1$ and $r \geq 1$. If $r \geq k$, then $S(n, a; p^k) = S(0, a; p^k)$ is a Ramanujan sum and equals to 0, since $(a, p) = 1$ and $k \geq 2$. If $1 \leq r < k$, noting

$$S(n, a; p^k) = \sum_{b \bmod p^k}^* e \left(\frac{p^r m \bar{b} + ab}{p^k} \right),$$

we have

$$S(n, a; p^k) = \sum_{y \bmod p^{k-r}}^* \sum_{x \bmod p^r} e \left(\frac{p^r m \overline{y + p^{k-r}x} + a(y + p^{k-r}x)}{p^k} \right).$$

Summing over x , we get

$$S(n, a; p^k) = \sum_{y \bmod p^{k-r}}^* e \left(\frac{p^r m y^{-1} + a y}{p^k} \right) \sum_{x \bmod p^r} e \left(\frac{a x}{p^r} \right) = 0,$$

which concludes the proof. \square

Let $\Re z$ denote the real part of a complex number z .

Lemma 3.4. *For $\gcd(a, q) = 1$ and $q = p^k$ with $k \geq 2$, we have*

$$S(n, a; q) = \begin{cases} 2 \left(\frac{l}{p} \right)^k q^{1/2} \Re \vartheta_q e \left(\frac{2l}{q} \right), & \text{if } \left(\frac{na}{p} \right) = 1, \\ 0, & \text{if } \left(\frac{na}{p} \right) = -1, \end{cases}$$

where $l^2 \equiv na \pmod{q}$, $\left(\frac{l}{p} \right)$ is the Legendre symbol, ϑ_q equals 1 if $q \equiv 1 \pmod{4}$ and i if $q \equiv 3 \pmod{4}$.

Proof. This is [9, Equation (12.39)]. \square

Lemma 3.5. *Suppose that $d, \mu \in \mathbb{N}$, $d \geq 300$, $\mu \geq d+1$, $\beta = \lfloor \mu/10 \rfloor + 1$, $f(X) = a_1 X + \dots + a_{d+1} X^{d+1} \in \mathbb{Z}[X]$. Let r be defined by the relation $P^r = p^\mu$ and $\mu \log p > 10^8 r d \log d$. Then if $1 \leq r \leq d/300$, there exists an absolute constant $c > 10^{-13}$, such that*

$$\left| \sum_{1 \leq x \leq P} e \left(\frac{f(x)}{p^\mu} \right) \right| \leq 3P^{1-c/r^2} + nR,$$

where R is the maximum number of solutions of the congruence

$$f^{(u)}(x) \equiv 0 \pmod{p^\beta}, \quad 1 \leq x \leq P,$$

for $25r \leq u \leq 27r$.

Proof. This is [14, Theorem 2]. \square

Lemma 3.6. *Suppose $f(X) = a_0 + a_1 X + \dots + a_d X^d \in \mathbb{Z}[X]$ with the coefficients satisfying $\gcd(a_0, \dots, a_d, m) = 1$. Let $\rho(f, m)$ be the number of solutions of the congruence*

$$f(x) \equiv 0 \pmod{m}.$$

Then for $d \geq 2$, we have

$$\rho(f, m) \leq c_d m^{1-1/d},$$

where $c_d = d/e + O(\log^2 d)$ with e being the base of the natural logarithm.

Proof. This is the main result of [12]. \square

Lemma 3.7. *Let Q, μ be positive integers, p be a prime number. Suppose $f(X) = a_0 + a_1X + \dots + a_dX^d \in \mathbb{Z}[X]$ with the coefficients satisfying $\gcd(a_0, \dots, a_d, p) = 1$. Then for the number of solutions $R(Q, p^\mu)$ of the congruence*

$$f(x) \equiv 0 \pmod{p^\mu}, \quad 1 \leq x \leq Q,$$

the estimate

$$R(Q, p^\mu) \ll d(Qp)^{1-1/d} + dQp^{-\mu/d}$$

holds, where the implied constant in \ll is absolute.

Proof. By Lemma 3.6, we have $R(p^\mu, p^\mu) \ll dp^{\mu-\mu/d}$. Then for $Q \geq p^\mu$,

$$R(Q, p^\mu) \leq R(p^\mu, p^\mu) \left(\frac{Q}{p^\mu} + 1 \right) \ll dQp^{-\mu/d}.$$

If $Q < p^\mu$, there exists a unique non-negative integer ω such that $p^{\omega-1} < Q \leq p^\omega$. It is clear that $\omega \leq \mu$ and $p^\omega \leq pQ$, which yields

$$R(Q, p^\mu) \leq R(Q, p^\omega) \leq R(p^\omega, p^\omega) \ll dp^{\omega-\omega/d} \leq d(Qp)^{1-1/d}.$$

Now the result follows from the above two estimates. \square

Remark 3.8. *We remark that for a fixed d , Konyagin and Steger [13] give stronger estimates on $R(Q, p^\mu)$ than that of Lemma 3.7, but we prefer to us to keep the dependence on d explicit. This maybe useful if one needs to derive a version of Theorem 1.2 with λ which is a slowly decreasing function of q .*

3.2. Concuding the proof. Let $\gcd(a, p) = 1$, $q = p^k$ with p an odd prime and $k \geq 2$ a positive integer. For a given $\lambda > 0$, we may suppose $10 \leq q^\lambda \leq N \leq q$ without loss of generality, and consider the upper bound of the sum

$$S(N) := \sum_{1 \leq n \leq N} S(mn, a; q).$$

Take

$$(3.1) \quad s := \left\lfloor \frac{\log N}{B \log p} \right\rfloor$$

with a sufficiently large constant $B > 0$ (depending on λ) and $T := \lfloor N/p^s \rfloor$.

Then Weil bound for Kloosterman sums (see [9, Corollary 11.12]) gives

$$S(N) = S(p^s T) + O(p^s d(q) q^{1/2}).$$

When $q^\lambda \leq N \leq q$, the O -term can be estimated trivially as

$$p^s d(q) q^{1/2} \ll q N^{1/B} \ll N q^{1/2 - (1-1/B)\lambda},$$

which is small enough, hence we only need to bound $S(p^s T)$. By Lemma 3.3, the sum over n with $p|n$ vanishes, thus

$$S(p^s T) = \sum_{\substack{1 \leq n \leq p^s T \\ \gcd(p, n) = 1}} S(mn, a; q).$$

Now we apply Lemma 3.4. Since there are two solutions for the quadratic congruence of l , it's necessary to note that the expression for Kloosterman sums doesn't depend on which solution we choose. Hence we may write

$$S(p^s T) = q^{1/2} \sum_{\substack{n \leq p^s T \\ \left(\frac{nma}{p}\right) = 1}} \sum_{l^2 \equiv nma \pmod{q}} \left(\frac{l}{p}\right)^k \Re \vartheta_q e\left(\frac{2l}{q}\right),$$

where $\sum_{l^2 \equiv nma \pmod{q}}$ means summing over the two solutions of the congruence $l^2 \equiv nma \pmod{q}$. Classify n by the remainder of $nma \pmod{p^s}$,

$$(3.2) \quad S(p^s T) = q^{1/2} \sum_{\substack{1 \leq \alpha < p^s \\ \left(\frac{\alpha}{p}\right) = 1}} \sum_{\substack{n \leq p^s T \\ nma \equiv \alpha \pmod{p^s}}} \sum_{l^2 \equiv nma \pmod{q}} \left(\frac{l}{p}\right)^k \Re \vartheta_q e\left(\frac{2l}{q}\right).$$

To solve the quadratic congruence in the inner sum, we use the following argument, which is similar to that in [10]. Since $(ma, q) = 1$, suppose $ma\xi \equiv 1 \pmod{q}$ and $\vartheta \equiv \xi \pmod{p^s}$ with $1 \leq \vartheta < p^s, s \geq 1$. From $nma \equiv \alpha \pmod{p^s}$, we have $n \equiv \vartheta\alpha \pmod{p^s}$, which implies that there exists $t \in \mathbb{Z}$, such that $n = \vartheta\alpha + p^s t$. Now we have

$$l^2 \equiv ma(\vartheta\alpha + p^s t) \equiv ma\vartheta\alpha(1 + \kappa p^s t) \pmod{q},$$

with $\vartheta\alpha\kappa \equiv 1 \pmod{q}$. Note that $ma\vartheta \equiv 1 \pmod{p}$, then $\left(\frac{ma\vartheta\alpha}{p}\right) = 1$. By Hensel's lemma, there exists $\omega \in \mathbb{Z}$, such that $\omega^2 \equiv ma\vartheta\alpha \pmod{q}$. Thus

$$l^2 \equiv nma \equiv \omega^2(1 + \kappa p^s t) \pmod{q}.$$

We remark that ω is determined by m, a, α, p^s and does not depend on n . Consider $1 + \kappa p^s t$ in the p -adic field \mathbb{Q}_p . By Taylor's expansion

(see [11, Chapter IV.1]), we have

$$(1 + \kappa p^s t)^{1/2} = 1 + \sum_{i=1}^{\infty} \binom{1/2}{i} \kappa^i p^{is} t^i,$$

for $s \geq 1$. Here the coefficients $\binom{1/2}{i} = \frac{1/2(1/2-1)\cdots(1/2-i+1)}{i!}$ with $i \geq 1$ happen to be p -adic integers, since p is an odd prime. Then we have

$$(1 + \kappa p^s t)^{1/2} \equiv \sum_{i=0}^{\lfloor k/s \rfloor} g(i) \kappa^i p^{is} t^i \pmod{p^k},$$

where $g(0) = 1$ and $g(i)$ with $1 \leq i \leq \lfloor k/s \rfloor$ are integers given by

$$(3.3) \quad g(i) \equiv \binom{1/2}{i} \pmod{p^k}, \quad 0 \leq g(i) < p^k.$$

Thus we get two solutions for the quadratic congruence of l in the inner sum of (3.2).

$$l \equiv \pm \omega f(t) \pmod{q},$$

where

$$(3.4) \quad f(t) := \sum_{i=0}^{\lfloor k/s \rfloor} g(i) \lambda^i p^{is} t^i.$$

Choosing the solution $l \equiv \omega f(t) \pmod{q}$ and noting that $f(t) \equiv g(0) \equiv 1 \pmod{p}$, we have

$$S(p^s T) = 2q^{1/2} \sum_{\substack{1 \leq \alpha < p^s \\ \left(\frac{\alpha}{p}\right)=1}} \left(\frac{\omega}{p}\right)^k \sum_{t \leq \frac{p^s T - \vartheta \alpha}{p^s}} \Re \vartheta_q e\left(\frac{2\omega f(t)}{q}\right),$$

which gives

$$S(p^s T) \leq 2q^{1/2} \sum_{\substack{1 \leq \alpha < p^s \\ \left(\frac{\alpha}{p}\right)=1}} \left| \sum_{t \leq \frac{p^s T - \vartheta \alpha}{p^s}} e\left(\frac{2\omega f(t)}{q}\right) \right|.$$

Recalling $1 \leq \vartheta < p^s$, we have

$$(3.5) \quad S(p^s T) \leq 2q^{1/2} \sum_{\substack{1 \leq \alpha < p^s \\ \left(\frac{\alpha}{p}\right)=1}} \left| \sum_{t \leq T} e\left(\frac{2\omega f(t)}{q}\right) \right| + O(q^{1/2} p^{2s}).$$

Since $B > 0$ in (3.1) is fixed and sufficiently large and $q^\lambda \leq N \leq q$, the contribution of the above O -term is

$$q^{1/2} p^{2s} \ll q^{1/2} N^{2/B} \ll N q^{1/2 - (1-2/B)\lambda},$$

which is small enough. Hence we only need to deal with the first term in (3.5). Denote the inner sum over t as

$$M := \sum_{t \leq T} e\left(\frac{2\omega f(t)}{q}\right).$$

Applying Lemma 3.5 to M , we obtain

$$(3.6) \quad |M| \leq 3T^{1-c/r^2} + dR.$$

Here $c > 10^{-13}$ is an absolute constant, r is given by $T^r = p^k$, $d := \lfloor k/s \rfloor$ is the degree of $f(t)$ and R is the maximal number of solutions of the congruences

$$(3.7) \quad f^{(u)}(x) \equiv 0 \pmod{p^\beta}, \quad 1 \leq x \leq T,$$

for $25r \leq u \leq 27r$, where $\beta := \lfloor k/10 \rfloor + 1$. Note that

$$(3.8) \quad T = \lfloor N/p^s \rfloor \geq \lfloor N^{1-1/B} \rfloor \geq \lfloor p^{(1-1/B)\lambda k} \rfloor \geq p^{(1-1/B)\lambda k/2}.$$

Recall $T^r = p^k$, then

$$(3.9) \quad r = \frac{k \log p}{\log T} \leq \frac{2}{(1-1/B)\lambda}.$$

Let \mathfrak{F}_1 denote the contribution of the term $3T^{(1-\frac{c}{r^2})}$ in (3.6) to $S(p^s T)$, then

$$\mathfrak{F}_1 \ll q^{1/2} p^s T^{1-c/r^2} \ll N q^{1/2} T^{-c/r^2} \ll N q^{1/2-\delta_1(\lambda)},$$

where

$$\delta_1(\lambda) := \frac{c\lambda^3(1-1/B)^3}{8}.$$

Now we estimate the contribution of dR in (3.6) to $S(p^s T)$. To this aim, we give the upper bound for $d = k/s$ first, which is

$$(3.10) \quad d \leq k/s = \frac{k}{\left\lfloor \frac{\log N}{B \log p} \right\rfloor} \leq \frac{k}{\lfloor \lambda k/B \rfloor} \leq \frac{2B}{\lambda},$$

provided

$$(3.11) \quad k \geq \frac{10B}{\lambda}.$$

Let R_u denote the number of solutions of the equation (3.7), then

$$R = \max_{25r \leq u \leq 27r} R_u.$$

From Lemma 3.7, we have

$$dR \ll d^2 \max_{25r \leq u \leq 27r} (pT)^{1-1/(d-u)} + d^2 T \max_{25r \leq u \leq 27r} p^{-(\beta - \text{ord}_p f^{(u)})/(d-u)},$$

which yields

$$(3.12) \quad dR \ll d^2 p \max_{25r \leq u \leq 27r} T^{1-1/d} + d^2 T \max_{25r \leq u \leq 27r} p^{-(\beta - \text{ord}_p f^{(u)})/d}.$$

Let \mathfrak{F}_2 denote the contribution of the first term on the right hand side to $S(p^s T)$. Then

$$\mathfrak{F}_2 \ll q^{1/2} d^2 p^{s+1} T^{1-1/d} \ll N q^{1/2} d^2 p T^{-1/d}.$$

Further, using the lower bound (3.8) of T and the upper bound (3.10) of d ,

$$\mathfrak{F}_2 \ll \frac{4B^2}{\lambda^2} q^{-(\lambda^2(B-1)/4+1/k)} \leq \frac{4B^2}{\lambda^2} q^{-\lambda^2(B-1)/8},$$

provided

$$(3.13) \quad k \geq \frac{8}{\lambda^2(B-1)}.$$

Now only the contribution of the second term in (3.12) to $S(p^s T)$ is left. Let's estimate the upper bound of $\text{ord}_p f^{(u)}$ for $25r \leq u \leq 27r$. Noting that $(\lambda, p) = 1$ in the definition (3.4) of $f(t)$, we have

$$\text{ord}_p(f^{(u)}) = \min_{\substack{u \leq i \leq d \\ g(i) \neq 0}} \left(\nu_p \left(g(i) \frac{i!}{(i-u)!} \right) + is \right).$$

We claim that if k is sufficiently large, then

$$g(i) \neq 0 \quad \text{and} \quad \nu_p(g(i)) = \nu_p \left(\binom{1/2}{i} \right),$$

for all $u \leq i \leq d$. To see this, recall

$$\binom{1/2}{i} = \frac{(-1)^{i-1} (2i-3)!}{2^{2i-2} i! (i-2)!} \quad \text{for } i \geq 3$$

which is an p -adic integer. Then an argument similar to that in the proof of Lemma 3.2 gives

$$\nu_p \left(\binom{1/2}{i} \right) = - \sum_{j=1}^{\frac{\log(2i-3)}{\log p}} \frac{1}{p^j} + E'(i, p),$$

with $|E'(i, p)| \leq \frac{3 \log(2i)}{\log p}$. Therefore, for $u \leq i \leq d$, we have

$$\nu_p \left(\binom{1/2}{i} \right) \leq \frac{3 \log(2d)}{\log p} \leq \frac{3 \log(4B/\lambda)}{\log 3},$$

which implies

$$\nu_p \left(\binom{1/2}{i} \right) \leq k - 1$$

provided

$$(3.14) \quad k \geq \frac{3 \log(4B/\lambda)}{\log 3} + 1.$$

Now our claim follows from the definition (3.3) of $g(i)$. Thus we can remove the condition $g(i) \neq 0$ for k satisfying the above condition and get

$$\text{ord}_p(f^{(u)}) = \min_{u \leq i \leq d} \left(\nu_p \left(\binom{1/2}{i} \frac{i!}{(i-u)!} \right) + is \right).$$

By Lemma 3.2, we have

$$\text{ord}_p(f^{(u)}) \leq \min_{u \leq i \leq d} \left(u \sum_{j=1}^{\infty} \frac{1}{p^j} + \frac{3 \log(2i)}{\log p} + is \right),$$

which yields

$$\text{ord}_p(f^{(u)}) \leq u \sum_{j=1}^{\infty} \frac{1}{p^j} + \frac{3 \log(2u)}{\log p} + us.$$

Hence, for every $25r \leq u \leq 27r$, we have an uniform bound

$$\text{ord}_p(f^{(u)}) \leq \frac{27r}{p-1} + 3 \log(54r) + 27rs.$$

Let \mathfrak{F}_3 denote the contribution of the second term in (3.12) to $S(p^s T)$, then

$$\mathfrak{F}_3 \ll (4B^2/\lambda^2) N q^{1/2} \max_{25r \leq u \leq 27r} p^{-(\beta - \text{ord}_p f^{(u)})/d}$$

by (3.5), (3.6) and (3.10). Recall $\beta = [k/10] + 1$ and by (3.10) again,

$$\frac{\beta - \text{ord}_p(f^{(u)})}{d} \geq \frac{\lambda}{2B} (k/10 - \text{ord}_p(f^{(u)})),$$

which gives

$$\frac{\beta - \text{ord}_p(f^{(u)})}{d} \geq \frac{\lambda}{2B} (k/10 - 3 \log(54r) - 54rs),$$

Note that $N \leq p^k$, then, recalling (3.1), we obtain

$$s = \left\lfloor \frac{\log N}{B \log p} \right\rfloor \leq \frac{\log p^k}{B \log p} \leq \frac{k}{B}.$$

From this and the upper bound (3.9) of r , we get

$$\frac{\beta - \text{ord}_p f^{(u)}}{d} \geq \frac{\lambda}{2B} \left(k/10 - \frac{108k}{\lambda(B-1)} - 3 \log \left(\frac{108}{\lambda(1-1/B)} \right) \right).$$

Taking $B = B(\lambda) > 0$ sufficiently large, such that

$$\frac{108}{\lambda(B-1)} \leq 1/20.$$

Then

$$\frac{\beta - \text{ord}_p f^{(u)}}{d} \geq \frac{\lambda}{2B} \left(k/20 - 3 \log \left(\frac{108}{\lambda(1-1/B)} \right) \right).$$

It follows that

$$\frac{\beta - \text{ord}_p f^{(u)}}{d} \geq \frac{\lambda k}{80B}$$

provided

$$(3.15) \quad k \geq 120 \log \left(\frac{108}{\lambda(1-1/B)} \right),$$

which yields

$$\mathfrak{F}_3 \ll (4B^2/\lambda^2) Nq^{1/2-\lambda k/(80B)}.$$

We now choose k_0 in such a way that for $k \geq k_0$ the conditions (3.11), (3.13), (3.14) and (3.15) are satisfied, this completes the proof of Theorem 1.2.

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